Solution to Math4230 Tutorial 12

1. Show that
   (a) For the function $f(x) = \|x\|$, we have

$$\partial f(x) = \begin{cases} x/\|x\| & \text{if } x \neq 0, \\ \{d | \|d\| \leq 1\} & \text{if } x = 0. \end{cases}$$

Solution:

(a) For $x \neq 0$, the function $f(x) = \|x\|$ is differentiable with $\nabla f(x) = x/\|x\|$, so that $\partial f(x) = \big\{\nabla f(x)\big\} = \big\{x/\|x\|\big\}$. Consider now the case $x = 0$. If a vector d is a subgradient of f at $x = 0$, then $f(z) \geq f(0) + d'z$ for all z, implying that

$$\|z\| \geq d'z, \qquad \forall\, z \in \Re^n.$$

By letting $z = d$ in this relation, we obtain $\|d\| \leq 1$, showing that $\partial f(0) \subset \big\{d \mid \|d\| \leq 1\big\}$.

On the other hand, for any $d \in \Re^n$ with $\|d\| \leq 1$, we have

$$d'z \leq \|d\| \cdot \|z\| \leq \|z\|, \qquad \forall\, z \in \Re^n,$$

which is equivalent to $f(0) + d'z \leq f(z)$ for all z, so that $d \in \partial f(0)$, and therefore $\big\{d \mid \|d\| \leq 1\big\} \subset \partial f(0)$.

2. Consider a proper convex function F of two vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. For a fixed $(\bar{x}, \bar{y}) \in \text{dom}(F)$, let $\partial_x F(\bar{x}, \bar{y})$ and $\partial_y F(\bar{x}, \bar{y})$ be the subdifferentials of the functions $F(\cdot, \bar{y})$ and $F(\bar{x}, \cdot)$ at $\bar{x}$ and $\bar{y}$, respectively.
   (a) Show that

$$\partial F(\bar{x}, \bar{y}) \subset \partial_x F(\bar{x}, \bar{y}) + \partial_y F(\bar{x}, \bar{y}),$$

   and give an example showing that the inclusion may be strict in general.
   (b) Assume that F has the form

$$F(x, y) = h_1(x) + h_2(y) + h(x, y)$$

   where h_1 and h_2 are proper convex functions, and h is convex, real-valued, and differentiable. Show that the formula of part (a) holds with equality.

Solution:

(a) We have $(g_x, g_y) \in \partial F(\bar{x}, \bar{y})$ if and only if

$$F(x,y) \geq F(\bar{x},\bar{y}) + g_x'(x-\bar{x}) + g_y'(y-\bar{y}), \qquad \forall\, x \in R^n,\ y \in R^m.$$

By setting $y = \bar{y}$, we obtain that $g_x \in \partial_x F(\bar{x},\bar{y})$, and by setting $x = \bar{x}$, we obtain that $g_y \in \partial_y F(\bar{x},\bar{y})$, so that $(g_x, g_y) \in \partial_x F(\bar{x},\bar{y}) \times \partial_y F(\bar{x},\bar{y})$.

For an example where the inclusion is strict, consider any function whose subdifferential is not a Cartesian product at some point, such as $F(x,y) = |x+y|$ at points $(\bar{x},\bar{y})$ with $\bar{x}+\bar{y}=0$.

(b) Since F is the sum of functions of the given form, we have

$$\partial F(\bar{x},\bar{y}) = \{(g_x,0) \mid g_x \in \partial h_1(\bar{x})\} + \{(0,g_y) \mid g_y \in \partial h_2(\bar{y})\} + \{\nabla h(\bar{x},\bar{y})\}$$

[the relative interior condition of the proposition is clearly satisfied]. Since

$$\nabla h(\bar{x},\bar{y}) = (\nabla_x h(\bar{x},\bar{y}), \nabla_y h(\bar{x},\bar{y})),$$

$$\partial_x F(\bar{x},\bar{y}) = \partial h_1(\bar{x}) + \nabla_x h(\bar{x},\bar{y}),$$

$$\partial_y F(\bar{x},\bar{y}) = \partial h_2(\bar{y}) + \nabla_y h(\bar{x},\bar{y}),$$

the result follows.

3. (Directional Derivative of Extended Real-Valued Functions)
Let $f : \mathbb{R}^n \longmapsto (-\infty, \infty]$ be a convex function, and let x be a vector in $\mathrm{dom}(f)$. Define

$$f'(x;y) = \inf_{\alpha>0} \frac{f(x+\alpha) - f(x)}{\alpha}, \quad y \in \mathbb{R}^n$$

Show the following:
(a) $f'(x;\lambda y) = \lambda f'(x;y)$ for all $\lambda \geq 0$ and $y \in \mathbb{R}^n$;
(b) $f'(x;\cdot)$ is a convex function;
(c) $-f'(x;-y) \leq f'(x;y)$ for all $y \in \mathbb{R}^n$

Solution:

(a) Since $f'(x;0) = 0$, the relation $f'(x;\lambda y) = \lambda f'(x;y)$ clearly holds for $\lambda = 0$ and all $y \in \Re^n$. Choose $\lambda > 0$ and $y \in \Re^n$. By the definition of directional derivative, we have

$$f'(x;\lambda y) = \inf_{\alpha>0} \frac{f\big(x+\alpha(\lambda y)\big) - f(x)}{\alpha} = \lambda \inf_{\alpha>0} \frac{f\big(x+(\alpha\lambda)y\big) - f(x)}{\alpha\lambda}.$$

By setting $\beta = \lambda\alpha$ in the preceding relation, we obtain

$$f'(x;\lambda y) = \lambda \inf_{\beta>0} \frac{f(x+\beta y) - f(x)}{\beta} = \lambda f'(x;y).$$

(b) Let (y_1, w_1) and (y_2, w_2) be two points in $\mathrm{epi}\big(f'(x;\cdot)\big)$, and let γ be a scalar with $\gamma \in (0,1)$. Consider a point (y_γ, w_γ) given by

$$y_\gamma = \gamma y_1 + (1-\gamma) y_2, \qquad w_\gamma = \gamma w_1 + (1-\gamma) w_2.$$

Since for all $y \in \Re^n$, the ratio

$$\frac{f(x+\alpha y) - f(x)}{\alpha}$$

is monotonically nonincreasing as $\alpha \downarrow 0$, we have

$$\frac{f(x+\alpha y_1)-f(x)}{\alpha} \le \frac{f(x+\alpha_1 y_1)-f(x)}{\alpha_1}, \qquad \forall\ \alpha, \alpha_1,\ \text{with } 0 < \alpha \le \alpha_1,$$

$$\frac{f(x+\alpha y_2)-f(x)}{\alpha} \le \frac{f(x+\alpha_2 y_2)-f(x)}{\alpha_2}, \qquad \forall\ \alpha, \alpha_2,\ \text{with } 0 < \alpha \le \alpha_2.$$

Multiplying the first relation by γ and the second relation by $1-\gamma$, and adding, we have for all α with $0 < \alpha \le \alpha_1$ and $0 < \alpha \le \alpha_2$,

$$\frac{\gamma f(x+\alpha y_1) + (1-\gamma) f(x+\alpha y_2) - f(x)}{\alpha} \le \gamma \frac{f(x+\alpha_1 y_1)-f(x)}{\alpha_1} + (1-\gamma)\frac{f(x+\alpha_2 y_2)-f(x)}{\alpha_2}.$$

From the convexity of f and the definition of y_γ, it follows that

$$f(x+\alpha y_\gamma) \le \gamma f(x+\alpha\gamma y_1) + (1-\gamma) f(x+\alpha y_2).$$

Combining the preceding two relations, we see that for all $\alpha \le \alpha_1$ and $\alpha \le \alpha_2$,

$$\frac{f(x+\alpha y_\gamma)-f(x)}{\alpha} \le \gamma \frac{f(x+\alpha_1 y_1)-f(x)}{\alpha_1} + (1-\gamma)\frac{f(x+\alpha_2 y_2)-f(x)}{\alpha_2}.$$

By taking the infimum over α, and then over α_1 and α_2, we obtain

$$f'(x; y_\gamma) \le \gamma f'(x; y_1) + (1-\gamma) f'(x; y_2) \le \gamma w_1 + (1-\gamma) w_2 = w_\gamma,$$

where in the last inequality we use the fact $(y_1, w_1), (y_2, w_2) \in \mathrm{epi}\big(f'(x;\cdot)\big)$ Hence the point (y_γ, w_γ) belongs to $\mathrm{epi}\big(f'(x;\cdot)\big)$, implying that $f'(x;\cdot)$ is a convex function.

(c) Since $f'(x; 0) = 0$ and $(1/2)y + (1/2)(-y) = 0$, it follows that

$$f'\big(x; (1/2)y + (1/2)(-y)\big) = 0, \qquad \forall\ y \in \Re^n.$$

By part (b), the function $f'(x;\cdot)$ is convex, so that

$$0 \le (1/2) f'(x; y) + (1/2) f'(x; -y),$$

and

$$-f'(x; -y) \le f'(x; y).$$